

Wave function of the Universe and Chern-Simons Perturbation Theory

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The Chern-Simons exact solution of four-dimensional quantum gravity with nonvanishing cosmological constant is presented in metric variable as the partition function of a Chern-Simons theory with nontrivial source. The perturbative expansion is given, and the wave function is computed to the lowest order of approximation for the Cauchy surface which is topologically a 3-sphere. The state is well-defined even at degenerate and vanishing values of the dreibein. Reality conditions for the Ashtekar variables are also taken into account; and remarkable features of the Chern-Simons state and their relevance to cosmology are pointed out.

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I. OVERVIEW

Soon after Ashtekar had recast and simplified the constraints of 4-d gravity [1-3], it became apparent the exponential of the Chern-Simons functional of the connection variable is an *exact* solution [4] of quantum gravity in the connection representation. Concurrent with these developments, Witten in his seminal paper [5] showed how quantum field theory with non-Abelian Chern-Simons action gave rise to 3-manifold and link invariants; and subsequent work by many authors have made the Chern-Simons theory one of the most successful topological theories to date. Much has also been learnt from studying quantum gravity with the new variables [6].

We begin with the observation that the Chern-Simons wave function is well-poised for the synthesis of these two strands of development; because the state expressed in metric variable, is roughly (this statement will be made precise in subsequent sections) the partition function of the Chern-Simons action with the conjugate variable as *source*.

Although loop and spin-network transforms of the Chern-Simons state have been considered before [7], without assuming mini-superspaces as in Refs. [4,8], the transformation to the more intuitive metric representation has been attempted, and computed for certain limits only fairly recently [9]. Yet the Chern-Simons state is arguably the most promising solution of quantum gravity with the new variables discovered so far. Despite being exact, it does not appear to suffer from the defects- among them the lack of long range correlation and sensible continuum limit [10]- that affect many known non-perturbative solutions. By virtue of being at the same time the Hamilton-Jacobi functional [4,11], it should lead to reasonable semi-classical correspondence. In fact, it is also cosmologically interesting because it is associated with the Cauchy data of constant curvature 3-manifolds [4,12,13] (this assertion is addressed briefly in Section IV). So even if the Chern-Simons wave function is but one solution, it may be quite useful to consider the simplifying assumption of the Chern-Simons state for quantum gravity as a “Quantum Cosmological Principle”, analogous to the classical assumption of “The Cosmological Principle” with its resultant Robertson-Walker metrics; and with regard to which we may want to consider perturbations. This differs from just doing mini-superspace quantum gravity because the Chern-Simons wave function is an exact solution of the full theory. Most remarkable of all its attributes is that when transformed to the metric (more precisely, the densitized triad) representation, the dimensionless coupling constant for the resultant Chern-Simons theory is $\kappa = \frac{3}{8\pi\lambda G}$. Not only does this offer an intriguing opportunity to entertain the cosmological constant, λ , as a derived function of κ and Newton’s constant, current astrophysical bounds place the value of κ at greater than 10^{120} . This implies the usual $\frac{1}{\sqrt{\kappa}}$ expansion [14] for Chern-Simons perturbation theory corresponds to a coupling of roughly 10^{-60} ; so even perturbative results for the Chern-Simons theory will be incredibly good, indeed more accurate than for any other known physical theory. However, if we were to entertain a Planck scale cosmological constant at, say, the inflationary era or earlier, alternative strategies would be required.

After briefly introducing Ashtekar’s variables and establishing the notations in Section II, the reality conditions are addressed in Section III. One of the challenges to the synthesis of the known results is how to reconcile the complex nature of Ashtekar’s variables for quantum gravity with ordinary real Chern-Simons perturbation theory. Remarkably, the key to the resolution lies in an inversion formula (to be discussed in Section III) discovered many years ago [4,12], which says that in transforming from the connection to the densitized triad representation, the integration is not to be

performed over all complex values of the connection but only along a contour parallel to the imaginary axis. It implies by a Wick rotation we can treat the integration connection variable as real in the Chern-Simons partition function. In Section IV we recall the Chern-Simons state and its semi-classical analog; and we adapt the known results to write down the perturbative expansion for the Chern-Simons wave function of quantum gravity in Section V. In the final section, some relevant topological issues are covered, including especially the effect of large gauge transformations, and the chosen normalization; and we end by computing the lowest order approximation of the wave function when the Cauchy manifold is topologically a 3-sphere.

II. PRELIMINARIES

Starting from real phase space variables $(\tilde{\sigma}^{ia}, k_{jb})$ with the non-trivial Poisson bracket¹

$$\{\tilde{\sigma}^{ia}(\vec{x}), k_{jb}(\vec{y})\}_{P.B.} = (8\pi G)\delta_j^i \delta_b^a \delta^3(\vec{x} - \vec{y}), \quad (2.1)$$

Ashtekar proposed the complex combination,

$$\begin{aligned} A_{ia} &\equiv ik_{ia} - \frac{1}{2}\epsilon_a{}^{bc}\omega_{ibc} \\ &= ik_{ia} + \frac{\delta F[\tilde{\sigma}]}{\delta \tilde{\sigma}^{ia}} \end{aligned} \quad (2.2)$$

and the densitized triad $\tilde{\sigma}^{ia} = \frac{1}{2}\epsilon^{abc}\tilde{\epsilon}^{ijk}e_{jb}e_{kc}$ as the fundamental pair of variables for 4-d gravity. ω_{ab} is the spin connection compatible with the dreibein, e_a i.e. $de_a + \omega_{ab} \wedge e^b = 0$; and modulo the constraints of general relativity, $e_j^a k_{ia}$ is the extrinsic curvature. Hence A_{ia} transforms as a complex $SO(3)$ connection, and its Poisson bracket with $\tilde{\sigma}^{ia}$ is just

$$\{\tilde{\sigma}^{ia}(\vec{x}), A_{jb}(\vec{y})\}_{P.B.} = i(8\pi G)\delta_j^i \delta_b^a \delta^3(\vec{x} - \vec{y}). \quad (2.3)$$

Since $-\frac{1}{2}\epsilon_a{}^{bc}\omega_{ibc}$ is the functional derivative of

$$F[\tilde{\sigma}] \equiv -\frac{1}{2} \int_M e^a \wedge de_a, \quad (2.4)$$

the Poisson bracket between two A variables vanishes. In other words, $F[\tilde{\sigma}]$ is the sought-after generating functional for the complex canonical transformation [15–17] to $(\tilde{\sigma}^{ia}, A_{jb})$. Remarkably, the seven constraints for 4-d gravity reduce to [1–3]

$$D_i^A \tilde{\sigma}^{ia} = 0, \quad \epsilon_{ijk} \tilde{\sigma}^{ia} \tilde{B}_a^j = 0 \quad (2.5)$$

and

$$\epsilon_{abc} \epsilon_{ijk} \tilde{\sigma}^{jb} \tilde{\sigma}^{kc} \left(\tilde{B}^{ia} + \frac{\lambda}{3} \tilde{\sigma}^{ia} \right) = 0; \quad (2.6)$$

with D^A denoting the covariant derivative with respect to A_{ia} , $\tilde{B}^{ia} \equiv \frac{1}{2}\tilde{\epsilon}^{ijk}F_{jka}$ the magnetic field, and λ the cosmological constant.

III. REALITY CONDITIONS, INNER PRODUCT, AND AN INVERSION FORMULA

The simplification of the constraints for 4-d space-times with Lorentzian signature was achieved at the cost of complexification. To recover the real phase space of general relativity, the simplest procedure -but not the only one [3]- is to impose $\tilde{\sigma}_{ia}^\dagger = \tilde{\sigma}^{ia}$ and $k_{ia}^\dagger = k_{ia}$ on the new phase space. The latter reality condition on k_{ia} translates into

¹We use natural units $\hbar = c = 1$, but retain G . Latin indices at the beginning of the alphabet denote $SO(3)$ indices, while those from i onwards are spatial indices.

$$A_{ia}^\dagger = -A_{ia} + 2 \frac{\delta F[\tilde{\sigma}]}{\delta \tilde{\sigma}^{ia}}. \quad (3.1)$$

Although somewhat complicated to implement in the A -representation (the measure becomes non-local then), it was pointed out in Refs. [16,4,12], and discussed in Ref. [3], that the generating functional $F[\tilde{\sigma}]$ is known explicitly; so the reality condition on A_{ia} is straightforward to uphold in the $\tilde{\sigma}$ -representation, and merely amounts to multiplying the original measure by $\exp(\frac{-2F[\tilde{\sigma}]}{8\pi G})$. In other words, if we realize the original commutation relations in the $\tilde{\sigma}$ -representation by

$$\hat{\tilde{\sigma}}^{ia} \Psi[\tilde{\sigma}] = \tilde{\sigma}^{ia} \Psi[\tilde{\sigma}], \quad \hat{k}_{ia} \Psi[\tilde{\sigma}] = (8\pi G) \frac{\delta}{i\delta \tilde{\sigma}^{ia}} \Psi[\tilde{\sigma}]; \quad (3.2)$$

then

$$\hat{A}_{ia} \mapsto (8\pi G) \exp\left(-\frac{F[\tilde{\sigma}]}{8\pi G}\right) \frac{\delta}{\delta \tilde{\sigma}^{ia}} \exp\left(\frac{F[\tilde{\sigma}]}{8\pi G}\right); \quad (3.3)$$

and it follows that the action of \hat{A} on $\exp(\frac{F[\tilde{\sigma}]}{8\pi G}) \Psi[\tilde{\sigma}]$, is just $\hat{A}_{ia} \mapsto (8\pi G) \frac{\delta}{\delta \tilde{\sigma}^{ia}}$. Moving over to the conjugate A -representation where $\hat{\tilde{\sigma}}^{ia} \Psi[A] = -(8\pi G) \frac{\delta}{\delta A_{ia}} \Psi[A]$, the relation between $\Psi[A]$ and $\exp(\frac{F[\tilde{\sigma}]}{8\pi G}) \Psi[\tilde{\sigma}]$ is just the usual Fourier transform [4,12,3]

$$\Psi[A] = \int D\tilde{\sigma} \exp\left[\frac{1}{8\pi G} (F[\tilde{\sigma}] - \int_M \tilde{\sigma}^{ia} A_{ia} d^3x)\right] \Psi[\tilde{\sigma}]. \quad (3.4)$$

To express $\Psi[\tilde{\sigma}]$ in terms of $\Psi[A]$, it is important to observe [4,12,3] the inversion is *not* by the naively integrating over *all* complex values of A , but only over a contour, C , *parallel to the imaginary axis*²

$$\Psi[\tilde{\sigma}] = \int_C DA \exp\left[\frac{1}{8\pi G} (-F[\tilde{\sigma}] + \int_M \tilde{\sigma}^{ia} A_{ia})\right] \Psi[A]. \quad (3.5)$$

This can be seen by decomposing $A = i\text{Im}(A) + \text{Re}(A)$ in Eq.(3.4) to give

$$\begin{aligned} \int D[\text{Im}(A)] \exp\left\{\frac{1}{8\pi G} \left[-F[\tilde{\sigma}] + i \int_M \tilde{\sigma} \cdot \text{Im}(A)\right]\right\} \Psi[A] &= \int D\tilde{\sigma}' \left\{ \exp\left[\frac{1}{8\pi G} (F[\tilde{\sigma}'] - F[\tilde{\sigma}] - \int_M \tilde{\sigma}' \cdot \text{Re}(A))\right] \cdot \right. \\ &\quad \left. \int D[\text{Im}(A)] \exp\left[i \int_M (\tilde{\sigma} - \tilde{\sigma}') \cdot \text{Im}(A)\right] \Psi[\tilde{\sigma}'] \right\}. \end{aligned} \quad (3.6)$$

It is the integration over $\text{Im}(A)$ which gives $\delta(\tilde{\sigma}' - \tilde{\sigma})$, and the resultant inversion formula of Eq.(3.5). Note that the inversion formula is not particular to the Chern-Simons state, but applicable in general; and suggests a strategy to circumvent the issue of the complex connection variable by transforming to the conjugate representation.

The factor $\exp(\frac{-F[\tilde{\sigma}]}{8\pi G})$ has been accounted for in Eq.(3.5), and the inner product for $\Psi[\tilde{\sigma}]$ is just [4,12,17]

$$\langle \Psi | \Psi \rangle = \int D\tilde{\sigma} \overline{\Psi[\tilde{\sigma}]} \Psi[\tilde{\sigma}]. \quad (3.7)$$

Since

$$\exp\left[\frac{1}{8\pi G} (-F[\tilde{\sigma}] + \int_M \tilde{\sigma}^{ia} A_{ia})\right] = \exp\left[\frac{1}{16\pi G} \left(\int e^a \wedge D_A e_a\right)\right], \quad (3.8)$$

it is explicitly diffeomorphism and gauge invariant [9].

²The inversion prescription in Refs. [4,12] is for integration along the real axis because the convention therein is $A = k + i\omega$. Our $A = ik - \omega$ is similar to the original variable in Ref. [1], and corresponds to integration parallel to the imaginary axis. If the integrand in Eq.(3.5) is holomorphic in A , then the contour can be deformed. However, the Chern-Simons functional is neither bounded from above nor below.

IV. THE CHERN-SIMONS STATE

In the connection representation with $\hat{\sigma}^{ia} \mapsto -(8\pi G)\frac{\delta}{\delta A_{ia}}$, we may express the operator

$$\begin{aligned}\hat{Q}^{ia} &\equiv \tilde{B}^{ia} + \frac{\lambda}{3}\hat{\sigma}^{ia} \\ &= -\exp(\kappa CS[A])\frac{\delta}{\kappa\delta A_{ia}}\exp(-\kappa CS[A]),\end{aligned}\quad (4.1)$$

in terms of the Chern-Simons functional

$$CS[A] = \frac{1}{2}\int_M (A^a \wedge dA_a + \frac{1}{3}\epsilon^{abc}A_a \wedge A_b \wedge A_c) \quad (4.2)$$

since $\frac{\delta CS[A]}{\delta A_{ia}} = \tilde{B}^{ia}$ if $\partial M = 0$. Here $\kappa \equiv \frac{3}{8\pi\lambda G}$ is the dimensionless parameter mentioned previously. For non-vanishing cosmological constant, the constraints of Eqs.(2.5) and (2.6) can be rewritten as

$$D_i^A \hat{Q}^{ia} \Psi[A] = 0, \quad \epsilon_{ijk} \hat{\sigma}_a^j \hat{Q}^{ia} \Psi[A] = 0 \quad (4.3)$$

and

$$\epsilon_{abc} \epsilon_{ijk} \hat{\sigma}^{jb} \hat{\sigma}^{kc} \hat{Q}^{ia} \Psi[A] = 0. \quad (4.4)$$

It follows, for this ordering, that a sufficient condition for an *exact* state $\Psi[A]$ to satisfy all the constraints of 4-d quantum gravity is that it is annihilated by \hat{Q}^{ia} [18,19]. It is easy to check the solution is

$$\Psi[A] = \mathcal{N} \exp(\kappa CS[A]), \quad \frac{\delta \mathcal{N}}{\delta A_{ia}} = 0; \quad (4.5)$$

with \mathcal{N} being a topological invariant, as indicated in Eq.(4.5). We shall refer to this state as the ‘‘Chern-Simons state’’. In Section VI we shall argue that \mathcal{N} is needed to provide a compensating topological factor under large gauge transformations. Since we are dealing with quantum gravity, we cannot rule out summing over topologically inequivalent closed 3-manifolds and bear in mind the general solution

$$\Psi[A] = \sum_{Top(M): \partial M=0} \mathcal{N}_M \exp(\kappa CS[A]_M). \quad (4.6)$$

It is most remarkable that G and λ has come together as the dimensionless coupling constant κ . Thus in the Chern-Simons perturbation theory to be discussed the cosmological constant may be considered as a derived quantity from G and κ . Note however that without further restrictions, the Chern-Simons functional is invariant under arbitrary small gauge transformations only if the 3-dimensional Cauchy surface M is without boundary ($\partial M = 0$); otherwise its functional derivative contains boundary effects, and the exponential of the Chern-Simons functional fails to solve the constraints.

We note in passing that in the connection representation, $\lambda = 0$ is a singular limit, since it changes the functional differential equation of the super-Hamiltonian constraint of Eq.(2.6) from third order to second. With regard to convergence, the perturbation series to be discussed is not any less well-defined for one sign of κ as for the other. This is also true for the coupling constant of a normal Chern-Simons theory. However, the theory at hand refers to gravity, and the coupling or cosmological constant is, at least semi-classically, not entirely independent of the topology of M . We note that $CS[A]$ is at the same time a Hamilton function for the classical Hamilton-Jacobi theory, since replacing $\tilde{\sigma}^{ia}$ by $-(8\pi G)\frac{\delta}{\delta A_{ia}}(\kappa CS[A]) = -\frac{3}{\lambda}\tilde{B}^{ia}$ solves all the classical constraints. This Cauchy data is also known as the Ashtekar-Renteln ansatz [20]. Under evolution, the ansatz describes classical Einstein manifolds with vanishing Weyl two-form i.e. $W_{AB} = R_{AB} - \frac{\lambda}{3}e_A \wedge e_B = 0$; with e_A denoting the vierbein in four dimensions, and R_{AB} the Riemann curvature two-form. Solutions with positive λ include the de Sitter manifold. On 3-d Cauchy surfaces, the Ashtekar-Renteln ansatz which should correspond to the semi-classical limit of the state, implies (with A as in Eq.(2.2))

$$iD_\omega k_a = 0, \quad R_{ab} + k_a \wedge k_b = \frac{\lambda}{3}e_a \wedge e_b, \quad (4.7)$$

on equating the real and imaginary parts. Solutions of the first equation include $k_a = \alpha e_a$ with constant α , which implies M is a 3-manifold with *constant* Riemannian curvature $R_{ab} = (\frac{\lambda}{3} - \alpha^2)e_a \wedge e_b$ and Ricci scalar $R = 2(\lambda - 3\alpha^2)$. Since two constant curvature surfaces with the same value of R are isometric [21], the simply-connected constant curvature 3-manifolds are exhausted by S^3, R^3 and H^3 with, respectively, $+, 0$ and $-$ curvature. Thus to the extent $\partial M = 0$ is required, the Chern-Simons state not only selects S^3 as the only closed alternative, but also implies the cosmological constant λ is positive since $R > 0$ for $\partial M = 0$; in which case $\alpha^2 < \frac{\lambda}{3}$. The actual 3-topology of our universe is not yet settled [22]; but to the extent that the Chern-Simons state is capable of describing our universe by association with the de Sitter phase [4,11] of inflationary expansion [23], of the three alternatives for simply-connected Robertson-Walker space-times, it would, barring topology changes during evolution after the inflationary phase, in fact be compatible only with the *closed* model for the 3-topology of our present universe.

V. THE WAVE FUNCTION AS A CHERN-SIMONS PERTURBATION SERIES

We shall now employ functional methods to expand the Chern-Simons state in the $\tilde{\sigma}$ -representation as a Chern-Simons perturbation series. Details on Chern-Simons perturbation theory can be found in Refs. [24–26]; and fortuitous factors allow the adaptation of these results to the case at hand, even though the connection variable for 4-d quantum gravity is complex.

From Eqs.(3.5) and (4.5), the Chern-Simons state in the $\tilde{\sigma}$ -representation is

$$\Psi[\tilde{\sigma}] = \mathcal{N} \int_C DA \exp\left\{\left(\frac{1}{16\pi G} \int_M e^a \wedge D_A e_a\right) + \kappa CS[A]\right\}. \quad (5.1)$$

We leave \mathcal{N} unspecified for the moment, and take the liberty of absorbing integration constants into it when the need arises. It will be brought up more concretely later on.

Writing the connection as³ $A = A^{(o)} + \frac{1}{\sqrt{\kappa}}a$ leads to

$$\Psi[\tilde{\sigma}] = \mathcal{N} \exp\left\{\left(\frac{1}{16\pi G} \int_M e^a \wedge D^{(o)} e_a\right) + \kappa CS[A^{(o)}]\right\} Z[\tilde{\sigma}, A^{(o)}], \quad (5.2)$$

with

$$Z[\tilde{\sigma}, A^{(o)}] = \int_C Da \exp\left\{\frac{1}{2} \int_M [a^a \wedge D^{(o)} a_a + 2\sqrt{\kappa}(F_{A^{(o)}}^a + \frac{\lambda}{6}\epsilon^{abc}e_b \wedge e_c) \wedge a_a + \frac{\epsilon^{abc}}{3\sqrt{\kappa}}a_a \wedge a_b \wedge a_c]\right\}. \quad (5.3)$$

This is true for *any* $A^{(o)}$. The question is how best to compute $Z[\tilde{\sigma}, A^{(o)}]$. It is customary to choose a stationary point to eliminate the linear term in a ; for the case at hand, this means $A^{(o)}$ satisfies $F_{A^{(o)}}^a + \frac{\lambda}{6}\epsilon^{abc}e_b \wedge e_c = 0$. However this leads to complications. Without the luxury of constraining e_a , solving for $A^{(o)}$ in terms of the dreibein implies a non-Hermitian quadratic term which is furthermore a complicated function of $\tilde{\sigma}$. Moreover, a linear term in a will anyway appear as a ghost-antighost-gauge connection coupling after gauge fixing.

Happily, the expression for $\Psi[\tilde{\sigma}]$ also suggests we treat $\tilde{\sigma}$ as the *source* for A ; and expand about a *flat connection* $F_{A^{(o)}}^a = 0$ i.e. $A^{(o)} = UdU^{-1}$ locally. It follows that

$$Z[\tilde{\sigma}, A^{(o)}] = \int_C Da \exp\left\{\frac{1}{2} \int_M [a^a \wedge D^{(o)} a_a + \frac{\epsilon^{abc}}{3\sqrt{\kappa}}a_a \wedge a_b \wedge a_c] + \frac{1}{8\pi G\sqrt{\kappa}} \int_M \tilde{\sigma}^{ia} a_{ia} d^3x\right\}. \quad (5.4)$$

The Chern-Simons functional is invariant under gauge and general coordinate transformations. To compute its generating functional through Gaussian integrals, gauge-fixing is required. This entails the introduction of an auxiliary metric g^{ij} which is *independent of*, and not to be confused with, the source $\tilde{\sigma}^{ia}$. If the gauge-fixing action is a Becchi-Rouet-Stora-Tyutin(BRST) variation, nilpotency of the BRST operation guarantees that the full action is BRST-invariant. To compute the generating functional, it is sufficient to take care of just the Yang-Mills invariance by adding the gauge fixing action

$$\int_M L_{\text{gauge-fixing}} = \int_M \delta_{BRST}[\bar{c}^a (D^{(o)} * a_a - * \xi b_a)]$$

³If κ is negative, we use $A^{(o)} + \frac{1}{\sqrt{|\kappa|}}a$ instead.

$$= - \int_M \left[\frac{1}{\sqrt{\kappa}} (b^a D^{(o)} * a_a + \bar{c}^a (D^{(o)} * [a, c])_a) + \bar{c}^a D^{(o)} * D^{(o)} c_a + * \xi b^a b_a \right]. \quad (5.5)$$

$*D^{(o)} * D^{(o)} c_a = \Delta c_a$ is just the action of the Laplacian twisted by the flat connection, Δ , acting on the ghost; and $*$ is the Hodge dual operator which depends on g^{ij} .

After gauge-fixing, the generating functional,

$$Z[\tilde{\sigma}, A^{(o)}] = \int_C da \int d\bar{c} \int dc \exp \left\{ \frac{1}{2} \int_M (a^a \wedge D_{A^{(o)}} a_a + \frac{\epsilon^{abc}}{3\sqrt{\kappa}} a_a \wedge a_b \wedge a_c - \frac{1}{\sqrt{\kappa}} \bar{c}^a (D^{(o)} * [a, c])_a - * \bar{c}^a \Delta c_a \right. \\ \left. + \frac{1}{8\pi G \sqrt{\kappa}} \int_M \tilde{\sigma}^{ia} a_{ia} d^3 x \right\} \int db \exp \left\{ \frac{-1}{\sqrt{\kappa}} \int_M (b^a D^{(o)} * a_a + * \xi b^a b_a) \right\}, \quad (5.6)$$

includes integrations over (c, \bar{c}) ghost-antighost; and the auxiliary field b . The Gaussian integral over b is just

$$\int db \exp \left\{ \frac{-1}{\sqrt{\kappa}} \int_M (b^a D^{(o)} * a_a + * \xi b^a b_a) \right\} = C_\xi \exp \left\{ -\frac{1}{4\kappa\xi} \int_M (*a^a) \wedge D^{(o)} * D^{(o)} * a_a \right\} \quad (5.7)$$

after integration by parts.

If we were forced to integrate over all complex values of a , the generating functional would not be well-defined. Fortunately, as was discussed in an earlier section, the integration contour C is only parallel to the imaginary axis. For convenience we choose the constant real part of the contour to be just $A^{(o)}$. As a result, the integration over A along C can be converted into integration over a along the *real* axis R by a simple Wick rotation, $a \mapsto -ia$. Hence the generating functional assumes the form

$$Z[\tilde{\sigma}, A^{(o)}] = \int_R da \int d\bar{c} \int dc \exp \left\{ -\frac{1}{2} \int_M [a^a \wedge (D^{(o)} - \frac{1}{2\kappa\xi} * D^{(o)} * D^{(o)} *) a_a + \frac{i\epsilon^{abc}}{3\sqrt{\kappa}} a_a \wedge a_b \wedge a_c \right. \\ \left. - \frac{i}{\sqrt{\kappa}} \bar{c}^a (D^{(o)} * [a, c])_a + * \bar{c}^a \Delta c_a] - \frac{i}{8\pi G \sqrt{\kappa}} \int_M \tilde{\sigma}^{ia} a_{ia} d^3 x \right\}. \quad (5.8)$$

Apart from the difference of a factor of i in the quadratic term in a , the expression is identical to the gauge-fixed generating functional of a Chern-Simons theory for a *real* $SU(2)$ connection. $J^{ia} \equiv \frac{1}{8\pi G \sqrt{\kappa}} \tilde{\sigma}^{ia}$ is manifestly the source for a_{ia} .

An upshot of gauge-fixing is that operator L in the quadratic term in a ,

$$\int \int dx dy a_{ia}(x) L^{ikab}(x, y) a_{kb}(y) \equiv -\frac{1}{2} \int \int dx dy a_{ia}(x) \delta(x - y) (\tilde{\epsilon}^{ijk} D^{(o)}_j - \frac{1}{2\kappa\xi} D^{(o)}_l g^{il} D^{(o)}_m \sqrt{g} g^{km})^{ab} a_{kb}(y), \quad (5.9)$$

is invertible⁴. The expedient choice of flat metric for g^{ij} yields the inverse as

$$L_{ijab}^{-1}(x, y) = \frac{2}{\Delta} \{ \epsilon_{ijk} D^{(o)k} - \frac{2\kappa\xi}{\Delta} D^{(o)}_i D^{(o)}_j \}_{ab} \delta(x, y) \quad (5.10)$$

The cubic term in a and the ghost-antighost-gauge interaction both carry a $\frac{1}{\sqrt{\kappa}}$ factor, and will therefore be addressed perturbatively by expanding in powers of $\frac{1}{\sqrt{\kappa}}$. To wit, we introduce sources $\bar{\eta}$ and η for c and \bar{c} respectively; and express the generating functional as

$$Z[J, A^{(o)}] = \left[\exp \left(\frac{1}{\sqrt{\kappa}} \int_M \epsilon^{abc} \left[\frac{i}{2} (D^{(o)i} \frac{\partial}{\partial \eta})_a \frac{\partial}{\partial J^{ib}} \frac{\partial}{\partial \bar{\eta}^c} + \frac{\tilde{\epsilon}^{ijk}}{3} \frac{\partial}{\partial J^{ia}} \frac{\partial}{\partial J^{jb}} \frac{\partial}{\partial J^{kc}} \right] \right) \right. \\ \left. \int d\bar{c} \int dc \exp \left[\int_M * (-\bar{c}^a \frac{\Delta}{2} c_a + \bar{\eta}^a c_a + \bar{c}^a \eta_a) \right] \right]$$

⁴Zero modes of Δ and L are taken into account carefully in Ref. [26]; and the effect of reducible connections on the Faddeev-Popov determinant is discussed in [27,28]. Here we assume the zero modes have been subtracted, and the operators are invertible.

$$\int_R da \exp \left(\left[-\frac{1}{2} \int_M \int_M dx dy a_{ia}(x) L^{ijab}(x, y) a_{jb}(y) \right] - i \int_M J^{ia} a_{ia} d^3x \right) \Big|_{\bar{\eta}=\eta=0}. \quad (5.11)$$

Again, the Gaussian integrals over ghost-antighost variables, and over real a , can be performed readily, leading to

$$Z[J, A^{(o)}] = \frac{\det(\Delta)}{\sqrt{\det(L)}} \left[\exp \left(\frac{1}{\sqrt{k}} \int_M \epsilon^{abc} \left[\frac{i}{2} (D^{(o)})^i \frac{\partial}{\partial \eta} \right)_a \frac{\partial}{\partial J^{ib}} \frac{\partial}{\partial \bar{\eta}^c} + \frac{\tilde{\epsilon}^{ijk}}{3} \frac{\partial}{\partial J^{ia}} \frac{\partial}{\partial J^{jb}} \frac{\partial}{\partial J^{kc}} \right] \right) \exp \left(\int_M * \bar{\eta}^a \frac{2}{\Delta} \eta_a \right) \exp \left(-\frac{1}{4} \int_M dx \int_M dy J^{ia}(x) L_{ijab}^{-1}(x, y) J^{jb}(y) \right) \right] \Big|_{\bar{\eta}=\eta=0}. \quad (5.12)$$

Note that the expression is defined for degenerate as well as non-degenerate dreibeins.

We shall be interested in computing the ratio $Z[J, A^{(o)}]/Z[J=0, A^{(o)}]$ rather than the absolute value of the generating functional. It is also permissible to assume the “Landau gauge” $\xi = 0$ after the gauge fixing. This corresponds to imposing the condition $D^{(o)} * a = 0$; and has the geometrical interpretation physical excitations are orthogonal (with respect to the metric g^{ij}) to gauge variations i.e. $\int_M (*a)^a \wedge (\delta_{BRST} A)_a^{(o)} = 0$. It also has the advantages of not introducing the external spurious scale ξ and avoiding infrared divergences [29–31].

Naively, in the Chern-Simons generating functional the auxiliary metric enters only through gauge fixing, so the stress tensor, which captures the dependence on g^{ij} , is formally a BRST commutator. However proof of full metric independence can be very involved. Ref. [32] discusses the metric dependence when regularization effects are also taken into account, and Ref. [24] contains a formal proof of metric independence. In this article, we do not investigate the metric dependence or independence beyond noting that the additional ingredient here is the source term which requires no reference to g^{ij} since $\tilde{\sigma}^{ia}$ is a densitized vector. Thus the metric dependence of the results is arguably no worse than for the usual Chern-Simons theory with source. This topic certainly merits more careful study, especially when the aim here is to reconstruct quantum geometrodynamics from quantum gauge dynamics. The present article should be regarded as a first attempt at systematically computing the Chern-Simons state in the metric representation via Chern-Simons perturbation theory.

VI. DISCUSSIONS AND FURTHER REMARKS

The limit for zero source is taken to be $Z[J=0, A^{(o)}] = Z[J, A^{(o)}]|_{J=0}$. As we shall see, for *vanishing* dreibein ($e_a = 0$) the wave function $\Psi[0] = \mathcal{N} \exp(\kappa CS[A^{(o)}]) Z[J=0, A^{(o)}]$ may be well-defined, even if classical 3d-manifolds with degenerate dreibeins were problematic. It is not entirely clear what “normalization factor” (\mathcal{N} of Eq.(5.1)) should be adopted; but we shall take the following considerations into account.

Unlike the *partition function*, $\int DA \exp(i\kappa CS[A])$, of the usual Chern-Simons theory, *no* quantization condition on κ for the *wave function* of Eq.(4.5) (or Eq.(5.1)) arises⁵. Strictly speaking, a *state* needs only to be invariant up to a phase under large gauge transformations - an example is the θ -vacuum [33]. Moreover, there is a difference of a factor of i ; so for the Chern-Simons state here large gauge transformations, which induces $CS[A] \mapsto CS[A] + 8\pi^2 n$, multiplies the wave function by $\exp(8\pi^2 n \kappa)$ instead of a phase factor. Without a compensating topological factor in \mathcal{N} , the naive Chern-Simons solution is therefore not even invariant up to a phase under large gauge transformations. Furthermore, since its magnitude can be made arbitrarily big by translation of the connection under large gauge transformations, the Chern-Simons functional is neither bounded from above nor below. These complications can however be neutralized by accommodating $\exp(\kappa CS[A^{(o)}])$ in the denominator of \mathcal{N} to cancel out the effect of large gauge transformations⁶. This means that we are in effect considering the state to be proportional to $\exp[\frac{1}{(16\pi G)} \int e^a \wedge D^{(o)} e_a] Z[\tilde{\sigma}, A^{(o)}]$. Since a transforms covariantly, $Z[\tilde{\sigma}, A^{(o)}]$ of Eq.(5.3) clearly implies the resultant state will be invariant under *both* small *and* large gauge transformations. Note also that although $CS[A]$ is not a topological invariant for arbitrary connections, the Chern-Simons functional of a flat connection is. In quantum field theories, \mathcal{N} usually carries the normalization factor $(Z[J=0])^{-1}$ (normalized generating functionals eliminate vacuum diagrams). For the case at hand we can opt to include $\exp(\kappa CS[A^{(o)}]) Z[J=0, A^{(o)}]$ in the denominator of \mathcal{N} .

⁵This does not imply there is no renormalization of κ . In normal Chern-Simons theory, quantum corrections results in a shift of the coupling constant. However, if we are computing the ratio $Z[J, A^{(o)}]/Z[J=0, A^{(o)}]$, any counter term, e.g. $\exp(\kappa' CS[A^{(o)}])$, should cancel out if it is identical for the theory with and without source.

⁶Ref. [9] uses a similar normalization, by dividing with the exponent of the winding number. By expanding about a flat connection, the topological invariant $\exp(\kappa CS[A^{(o)}])$ arises naturally in our present discussion.

The absolute value of the ratio of the determinants in Eq.(5.12) is the square root of the Reidemeister-Ray-Singer analytic torsion, $\tau_{(o)}$, of the flat connection $A^{(o)}$ [34]. Since the Laplacian operator is positive-definite, the ratio, including the phase which comes only from L , is expressible as

$$\frac{\det(\Delta)}{\sqrt{\det(L)}} = \tau_{(o)}^{\frac{1}{2}} \exp(-i\pi\eta_L[A^{(o)}]), \quad (6.1)$$

in which the spectral asymmetry of L is accounted for by a suitably regularized eta-invariant [5]. This ratio can be made independent of the background metric by adding a phase, which is the linear combination of the gravitational Chern-Simons functional and the eta-invariant of L coupled to g^{ij} and the trivial connection [5]. However, if we are interested in $Z[J, A^{(o)}]/Z[J=0, A^{(o)}]$, the ratio of the determinants, together with its framing dependence on g^{ij} , *cancels out* since it appears in both the numerator and denominator.⁷

Taking the above considerations into account, and for concreteness, we choose \mathcal{N} so that the wave function

$$\Psi[J, A^{(o)}] = \exp\left(\frac{1}{16\pi G} \int e^a \wedge D^{(o)} e_a\right) \frac{Z[J, A^{(o)}]}{Z[J=0, A^{(o)}]}, \quad (6.2)$$

with $J^{ia} \equiv \frac{1}{16\pi G\sqrt{\kappa}} \epsilon^{abc} \tilde{\epsilon}^{ijk} e_{jb} e_{kc}$, is invariant under both small and large gauge transformations.⁸ This implies at vanishing dreibein $\Psi[0] = 1$.

As an illustration, consider $M = S^3$ topologically; $A^{(o)}$ is hence gauge equivalent to the trivial connection. Adopting a flat auxiliary metric and the Landau gauge, the resultant generating functional is

$$\begin{aligned} Z[J, A^{(o)}] = & \frac{\det(\Delta)}{\sqrt{\det(L)}} \left[\exp\left(\frac{1}{\sqrt{k}} \int \epsilon^{abc} \left[\frac{i}{2} (\partial^i \frac{\partial}{\partial \eta})_a \frac{\partial}{\partial J^{ib}} \frac{\partial}{\partial \bar{\eta}^c} + \frac{\tilde{\epsilon}^{ijk}}{3} \frac{\partial}{\partial J^{ia}} \frac{\partial}{\partial J^{jb}} \frac{\partial}{\partial J^{kc}}\right]\right) \right. \\ & \left. \exp\left\{\int_M \bar{\eta}^a \frac{2}{\Delta} \eta_a\right\} \exp\left\{-\frac{1}{2} \int_M \epsilon_{ijk} J^{ia} \frac{D^{(o)j}}{\Delta} J^k\right\}\right] \Bigg|_{\bar{\eta}=\eta=0}. \end{aligned} \quad (6.3)$$

The propagators for ghost-antighost, and for a ; and the 3-point vertices can be read off easily. On recalling the simple identities

$$\Delta\left(\frac{1}{|x-y|}\right) = -4\pi\delta(x-y), \quad \partial^i \frac{1}{|x-y|} = -\frac{(x-y)^i}{|x-y|^3}, \quad (6.4)$$

the two-point function⁹ is [35]

$$\langle a_{ia}(x) a_{jb}(y) \rangle_J = -\delta_{ab} \epsilon_{ijk} \frac{(x-y)^k}{8\pi|x-y|^3}. \quad (6.5)$$

Standard Feynman diagram techniques in quantum field theory can be applied to compute the perturbative expansion, in $\frac{1}{\sqrt{\kappa}}$, of the exponential operator with functional derivatives in Eq.(6.3). To lowest order, we find

$$\frac{Z[\tilde{\sigma}, A^{(o)}]}{Z[\tilde{\sigma}=0, A^{(o)}]} \approx \exp\left\{\frac{-1}{2^9 \kappa G^2 \pi^3} \int dx \int dy \epsilon_{ijk} \tilde{\sigma}^{ia}(x) \frac{(x-y)^j}{|x-y|^3} \tilde{\sigma}^{ka}(y)\right\}. \quad (6.6)$$

exhibiting inverse square law long range correlation. Note also that $\lim_{x \rightarrow y} \epsilon_{ijk} \tilde{\sigma}^{ia}(x) \tilde{\sigma}^{ka}(y) = 0$ curbs the singular behaviour from coincident points. However, we observe that Eq.(6.6) is not explicitly gauge invariant; and the often cited propagator of Eq.(6.5) does not transform in an explicitly covariant manner. Following a suggestion of Schwinger

⁷If M supports more than one distinct flat connection, we should sum, or integrate, over each contribution in Eq.(5.2) and may want to consider $\mathcal{N}^{-1} = \sum_{(o)} \exp(\kappa CS[A^{(o)}]) Z[J=0, A^{(o)}]$.

⁸There remains of course the freedom to multiply $\Psi[J, A^{(o)}]$ by a true topological invariant which could come in useful when we consider its normalization for the integration over $\tilde{\sigma}$ in the norm.

⁹This “propagator” for the Chern-Simons theory should not be confused with the expectation value, $\langle \Psi | \hat{A}_{ia}(x) \hat{A}_{jb}(y) | \Psi \rangle = -\frac{1}{\kappa} \int d\tilde{\sigma} \left[\frac{\delta \Psi[\tilde{\sigma}]}{\delta J^{ia}(x)} \right] \frac{\delta \Psi[\tilde{\sigma}]}{\delta J^{jb}(y)}$ in quantum gravity. Rather, the Chern-Simons “propagator” is a computational device for the latter.

many years ago [36], the situation can be rectified with a phase factor, $P(\exp \int_y^x A^{(o)})$, along a path connecting x and y . It is to be noted this factor here is *independent* of the path, no matter how far separated the points x and y are, because $A^{(o)}$ is a *flat* connection and every closed loop in S^3 is contractible. The justification is that careful computations with $A^{(o)} = U dU^{-1}$ in Eqs.(5.9) and (6.3), indeed produces an extra factor of $U(x)U^{-1}(y) = P(\exp \int_x^y A^{(o)})$ which takes the place of the δ_{ab} in the propagator in Eq.(6.5).

Since $A^{(o)}$ is flat, the factor $\int_M e^a \wedge D^{(o)} e_a$ can also be written in terms of Chern-Simons functionals and the volume of M as

$$\frac{1}{2l_p^2} \int_M e^a \wedge D^{(o)} e_a = CS[A^{(o)} + (e/l_p)] - CS[A^{(o)}] - \frac{1}{6l_p^3} \int_M \epsilon^{abc} e_a \wedge e_b \wedge e_c, \quad (6.7)$$

with the first term denoting the Chern-Simons functional of $A_a^{(o)}$ shifted by e_a divided by the Planck length $l_p = \sqrt{G}$. Combining these previous expressions, to the lowest order,

$$\begin{aligned} \Psi[\tilde{\sigma}] \approx & \exp \left\{ \frac{1}{8\pi} (CS[A^{(o)} + (e/l_p)] - CS[A^{(o)}] - \frac{V_M}{l_p^3}) \right\} \cdot \\ & \exp \left\{ \frac{-1}{29\kappa l_p^4 \pi^3} \int dx \int dy \epsilon_{ijk} \tilde{\sigma}^{ia}(x) [P(\exp \int_y^x A^{(o)})]_{ab} \tilde{\sigma}^{kb}(y) \frac{(x-y)^j}{|x-y|^3} \right\}. \end{aligned} \quad (6.8)$$

There is still a long way to go before the quantum fluctuations contained in Eq.(6.8) can be tested, say, in terms of the cosmic microwave radiation. At the very least we do not expect a finite norm for $\Psi[\tilde{\sigma}]$. The non-normalizability of the Chern-Simons state has been demonstrated explicitly in a mini-superspace model [37], and in a certain limit after gauge fixing [9]. In fact, the divergence of the norm is to be expected on general grounds [38]. Some have argued until we confront “the problem of time” [39], a probability amplitude interpretation for $\Psi[\tilde{\sigma}]$ does not make sense; since integrating over $\tilde{\sigma}$ amounts to summing over all physical “times”. Thus naive “non-normalizability” of a solution of the constraints does not imply spuriousness. Deciding what a wave function of the universe really means forces us to confront deep conceptual and interpretive issues [39]. Being exact despite its simplicity, computable systematically in familiar metric terms, semi-classically relevant and cosmologically interesting; the Chern-Simons wave function can be a valuable proving ground for these quantum gravity issues and their proposed resolutions.

In the “slow-roll” stage of inflationary scenarios [40], the effective cosmological constant of the de Sitter expansion phase can be estimated by $\frac{\lambda_{eff}}{(8\pi G)} \sim \sigma_{S.B.} T_{GUT}^4$, where $\sigma_{S.B.}$ denotes the Stefan-Boltzmann constant. This corresponds to an effective Chern-Simons coupling of $\frac{1}{\sqrt{\kappa_{eff}}} \sim 10^{-7} - 10^{-5}$ if the grand unification energy scale is taken to be $10^{15} - 10^{16}$ GeV. So Chern-Simons perturbation theory should be applicable during this stage. Moreover, since this grand unification scale is several orders below the Planck regime, semi-classical gravity is expected to dominate although it may still be necessary and desirable to include quantum gravity fluctuations. Thus the perturbative Chern-Simons wave function discussed here may be most pertinent to the inflationary period of our universe.

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